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Critical value computation for H_∞-Riccati difference equations

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Abstract

In designing finite horizon discrete time H_∞ controllers, the associated H_∞-Riccati difference equations must be solved. But the Riccati equation has a non-negative solution only when γ^{-2} is small enough. So it is important to get the upper bound of the parameter, i.e., the critical value that ensures the existence of the solution to the Riccati equation. The solution sequence of the Riccati difference equation can be constructed by the conjoined basis of an associated linear Hamiltonian difference system. Based on this expression and the Hamiltonian difference system eigenvalue theorems, the equivalence between the critical value and the first order eigenvalue of the linear Hamiltonian difference system is presented. Since the critical value is also shown to be the fundamental eigenvalue of a generalized Rayleigh quotient, an extended form of Wittrick–Williams algorithm is presented to search this value.

1. Introduction

The existence of a full information discrete time H_∞ controller depends on the existence of a non-negative solution to the associated Riccati difference equation and a matrix inequality [1]. Similar to Riccati differential equations of continuous time H_∞ control and filtering problems, the Riccati difference equation has a non-negative solution only for a small enough parameter γ^{-2}. Therefore it is important to determine the upper bound of γ^{-2}, namely the critical value γ_{cr}^{-2}, which ensures the existence of solutions of the Riccati difference equation. According to eigenvalue theorems of discrete linear Hamiltonian systems [2], the solution of the Riccati difference equation tends to infinity at the initial point when γ^{-2} is the fundamental eigenvalue of the Hamiltonian

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difference system. Which indicates that the fundamental eigenvalue γ_1^{-2} equals to γ_{cr}^{-2}. It is also presented in the paper that γ_{cr}^{-2} is equivalent to the fundamental eigenvalue of a generalized matrix eigenvalue problem. The argument is based on the concept of the generalized Rayleigh quotient with two kinds of variables and the discrete Legendre transformation.

For continuous time H_∞ control systems, on the basis of the correspondence between γ_{cr}^{-2} and the fundamental eigenvalue of a generalized Rayleigh quotient, the extended form of Wittrick–Williams (W–W) algorithm [3] is proposed to calculate γ_{cr}^{-2} [4]. The W–W algorithm is derived in Ref. [5] based on the Rayleigh quotient and used in structural mechanics, then a mathematical proof of this algorithm is presented in Ref. [6]. To overcome numerical ill-conditioning in computation, the extended W–W algorithm is proposed in Ref. [3] based on the analogy between structural mechanics and LQ control [7]. In this paper, another version of the extended algorithm is presented for the critical value computation of the H_∞ difference Riccati equations.

Section 2 briefly describes the full information discrete time H_∞ control problem and the associated Riccati difference equation according to Refs. [1,8]. Section 3 presents the relationship of the critical value γ_{cr}^{-2}, the first order eigenvalue of the Hamiltonian difference system and the generalized Rayleigh quotient. With the discrete Legendre transformation, an equivalent generalized matrix eigenvalue problem is also presented for the purpose of deriving the extended W–W algorithm. And the algorithm is formulated in Section 4. Section 5 summarizes the computational results of numerical examples.

2. Riccati difference equation of discrete H_∞ control

Consider the following linear discrete time system:

$$\mathbf{x}_{k+1} = \mathbf{A}_k\mathbf{x}_k + \mathbf{B}_k\mathbf{u}_k + \mathbf{D}_k\mathbf{w}_k, \quad \mathbf{x}_0 = 0, \tag{2.1}$$

$$\mathbf{z}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{S}_k\mathbf{u}_k, \tag{2.2}$$

where $k \in [0, N-1]$, state vector $\mathbf{x}_k \in R^n$, disturbance vector $\mathbf{w}_k \in R^l$, control vector $\mathbf{u}_k \in R^m$, output vector $\mathbf{z}_k \in R^p$. $\mathbf{A}_k$, $\mathbf{B}_k$, $\mathbf{D}_k$, $\mathbf{H}_k$ and $\mathbf{S}_k$ are matrices with appropriate dimensions. It is also assumed that $\mathbf{S}_k^{\mathrm{T}}[\mathbf{H}_k\ \mathbf{S}_k] = [\mathbf{0}\ \mathbf{I}]$, $\mathbf{H}_k^{\mathrm{T}}\mathbf{H}_k = \mathbf{Q}_k$. The object of H_∞ control is to find a control strategy $\{\mathbf{u}_k\}$ in the square summable space $\mathscr{L}_2[0, N-1]$ such that

$$\frac{1}{2}\sum_{k=0}^{N-1}\mathbf{z}_k^{\mathrm{T}}\mathbf{z}_k + \frac{1}{2}\mathbf{x}_N^{\mathrm{T}}\mathbf{Q}_f\mathbf{x}_N < \frac{1}{2}\gamma^2\sum_{k=0}^{N-1}\mathbf{w}_k^{\mathrm{T}}\mathbf{w}_k, \tag{2.3}$$

where $\mathbf{w} \in \mathscr{L}_2[0, N-1]$, $\gamma > 0$, $\mathbf{Q}_f$ is a symmetric positive-definite matrix. The feedback control law is

$$\mathbf{u}_k = \mathbf{K}\mathbf{x}_k \tag{2.4}$$

in which $\mathbf{K}$ is the feedback gain matrix. A solution to the problem was given by Basar and Bernhard [1], Yaesh and Shaked [8,9] as follows.

There exists a unique feedback controller that guarantees (2.4) if and only if the condition

$$\gamma^2\mathbf{I} - \mathbf{D}_k^{\mathrm{T}}\mathbf{M}_{k+1}\mathbf{D}_k > 0, \quad k \in [0, N-1] \tag{2.5}$$

holds, where the non-negative definite matrices sequence $\mathbf{M}_{k+1}$ is generated by the Riccati difference equation

$$\mathbf{M}_k = \mathbf{Q}_k + \mathbf{A}_k^{\mathrm{T}}\mathbf{M}_{k+1}[\mathbf{I} + (\mathbf{B}_k\mathbf{B}_k^{\mathrm{T}} - \gamma^{-2}\mathbf{D}_k\mathbf{D}_k^{\mathrm{T}})\mathbf{M}_{k+1}]^{-1}\mathbf{A}_k, \quad \mathbf{M}_N = \mathbf{Q}_f. \tag{2.6}$$

The state feedback control law and the disturbance are given by

$$\mathbf{u}_k = -\mathbf{B}_k^{\mathrm{T}}\mathbf{M}_{k+1}\mathbf{Z}_k^{-1}\mathbf{A}_k\mathbf{x}_k, \tag{2.7a}$$

$$\mathbf{w}_k = \gamma^{-2}\mathbf{D}_k^{\mathrm{T}}\mathbf{M}_{k+1}\mathbf{Z}_k^{-1}\mathbf{A}_k\mathbf{x}_k, \tag{2.7b}$$

where

$$\mathbf{Z}_k = \mathbf{I} + (\mathbf{B}_k\mathbf{B}_k^{\mathrm{T}} - \gamma^{-2}\mathbf{D}_k\mathbf{D}_k^{\mathrm{T}})\mathbf{M}_{k+1}. \tag{2.8}$$

Since Eq. (2.6) has a solution on finite horizon $[0, N-1]$ only when γ^{-2} is small enough, it is important to determine the upper bound γ_{cr}^{-2} of γ^{-2} first.

The results of the time-invariant system control are similar, except that system matrices $\mathbf{A}_k$, $\mathbf{B}_k$, $\mathbf{D}_k$, $\mathbf{H}_k$ and $\mathbf{S}_k$ are time invariant as shown in the following system:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{D}\mathbf{w}_k, \tag{2.9}$$

$$\mathbf{z}_k = \mathbf{H}\mathbf{x}_k + \mathbf{S}\mathbf{u}_k. \tag{2.10}$$

There exists a feedback controller if and only if the condition

$$\gamma^2\mathbf{I} - \mathbf{D}^{\mathrm{T}}\mathbf{M}_{k+1}\mathbf{D} > 0 \tag{2.11}$$

holds, where $\mathbf{M}_{k+1}$ is the solution of the Riccati recurrence equation

$$\mathbf{M}_k = \mathbf{Q} + \mathbf{A}^{\mathrm{T}}\mathbf{M}_{k+1}[\mathbf{I} + (\mathbf{B}\mathbf{B}^{\mathrm{T}} - \gamma^{-2}\mathbf{D}\mathbf{D}^{\mathrm{T}})\mathbf{M}_{k+1}]^{-1}\mathbf{A}, \quad \mathbf{M}_N = \mathbf{Q}_f \tag{2.12}$$

in which $\mathbf{Q} = \mathbf{H}^{\mathrm{T}}\mathbf{H}$. The results can also be extended to infinite horizon case, that the controller of an infinite horizon H_∞ control problem exists if and only if

$$\gamma^2\mathbf{I} - \mathbf{D}^{\mathrm{T}}\mathbf{M}\mathbf{D} > 0, \tag{2.13}$$

where $\mathbf{M}$ is the solution of the algebraic Riccati equation

$$\mathbf{M} = \mathbf{Q} + \mathbf{A}^{\mathrm{T}}\mathbf{M}[\mathbf{I} + (\mathbf{B}\mathbf{B}^{\mathrm{T}} - \gamma^{-2}\mathbf{D}\mathbf{D}^{\mathrm{T}})\mathbf{M}]^{-1}\mathbf{A}. \tag{2.14}$$

3. Critical value of the H_∞-Riccati difference equation

3.1. Critical value and Hamiltonian difference system eigenvalue

The solution sequence of the Riccati difference equation (2.6) can also be constructed by the conjoined basis of the following Hamiltonian difference equation:

$$\mathbf{x}_{k+1} = \mathbf{A}_k\mathbf{x}_k + (\gamma^{-2}\mathbf{D}_k\mathbf{D}_k^{\mathrm{T}} - \mathbf{B}_k\mathbf{B}_k^{\mathrm{T}})\boldsymbol{\lambda}_{k+1}, \tag{3.1a}$$

$$\boldsymbol{\lambda}_k = \mathbf{H}_k^{\mathrm{T}}\mathbf{H}_k\mathbf{x}_k + \mathbf{A}_k^{\mathrm{T}}\boldsymbol{\lambda}_{k+1} \tag{3.1b}$$

with boundary conditions

$$\mathbf{x}_0 = \mathbf{0}, \quad \boldsymbol{\lambda}_N = \mathbf{Q}_f \mathbf{x}_N. \tag{3.2}$$

The equivalent standard form of (3.2) is

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_f \end{bmatrix} \begin{Bmatrix} -\mathbf{x}_0 \\ \mathbf{x}_N \end{Bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\lambda}_0 \\ \boldsymbol{\lambda}_N \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}. \tag{3.3}$$

And the general form of boundary conditions of Hamiltonian difference system presented in Ref. [2] is

$$\begin{bmatrix} -\mathbf{R}_0^{\#} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_N^{\#} \end{bmatrix} \begin{Bmatrix} -\mathbf{x}_0 \\ \mathbf{x}_N \end{Bmatrix} + \begin{bmatrix} \mathbf{R}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_N \end{bmatrix} \begin{Bmatrix} \boldsymbol{\lambda}_0 \\ \boldsymbol{\lambda}_N \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{3.4}$$

in which the $n \times n$ matrices $\mathbf{R}_0$, $\mathbf{R}_0^{\#}$, $\mathbf{R}_N$, $\mathbf{R}_N^{\#}$ satisfy

$$\operatorname{rank}\begin{bmatrix} \mathbf{R}_0 \\ \mathbf{R}_0^{\#} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \mathbf{R}_N \\ \mathbf{R}_N^{\#} \end{bmatrix} = n,$$

$$\mathbf{R}_0 \mathbf{R}_0^{\#^{\mathrm{T}}} = \mathbf{R}_0^{\#} \mathbf{R}_0^{\mathrm{T}}, \quad \mathbf{R}_N \mathbf{R}_N^{\#^{\mathrm{T}}} = \mathbf{R}_N^{\#} \mathbf{R}_N^{\mathrm{T}}.$$

The general form of boundary condition (3.4) is for the use of theorems of Ref. [2] later.

The conjoined basis $(\mathbf{X}, \boldsymbol{\Lambda})$ of Hamiltonian difference equation (3.1) is composed of the $n \times n$ matrices sequence $\mathbf{X}_k$ and $\boldsymbol{\Lambda}_k$ (instead of the vectors $\mathbf{x}_k$, $\boldsymbol{\lambda}_k$), which are solved from the matrix form of Eq. (3.1), i.e.,

$$\mathbf{X}_{k+1} = \mathbf{A}_k \mathbf{X}_k + (\gamma^{-2}\mathbf{D}_k\mathbf{D}_k^{\mathrm{T}} - \mathbf{B}_k\mathbf{B}_k^{\mathrm{T}})\boldsymbol{\Lambda}_{k+1}, \tag{3.5a}$$

$$\boldsymbol{\Lambda}_k = \mathbf{H}_k^{\mathrm{T}}\mathbf{H}_k\mathbf{X}_k + \mathbf{A}_k^{\mathrm{T}}\boldsymbol{\Lambda}_{k+1} \tag{3.5b}$$

and satisfying

$$\operatorname{rank}\begin{bmatrix} \mathbf{X}_k^{\mathrm{T}} \\ \boldsymbol{\Lambda}_k^{\mathrm{T}} \end{bmatrix} = n, \tag{3.6a}$$

$$\mathbf{X}_k^{\mathrm{T}}\boldsymbol{\Lambda}_k = \boldsymbol{\Lambda}_k^{\mathrm{T}}\mathbf{X}_k. \tag{3.6b}$$

Let $\mathbf{P}_k = \boldsymbol{\Lambda}_k\mathbf{X}_k^{-1}$, Eq. (3.5) can be rewritten as

$$\mathbf{P}_k = \mathbf{H}_k^{\mathrm{T}}\mathbf{H}_k + \mathbf{A}_k^{\mathrm{T}}\mathbf{P}_{k+1}(\mathbf{X}_k\mathbf{X}_{k+1}^{-1})^{-1}, \tag{3.7a}$$

$$\mathbf{I} = \mathbf{A}_k\mathbf{X}_k\mathbf{X}_{k+1}^{-1} + (\gamma^{-2}\mathbf{D}_k\mathbf{D}_k^{\mathrm{T}} - \mathbf{B}_k\mathbf{B}_k^{\mathrm{T}})\mathbf{P}_{k+1}. \tag{3.7b}$$

Since condition (2.5) implies the non-singularity of $\mathbf{I} + (\mathbf{B}_k\mathbf{B}_k^{\mathrm{T}} - \gamma^{-2}\mathbf{D}_k\mathbf{D}_k^{\mathrm{T}})\mathbf{P}_{k+1}$ [1], Eqs. (3.7a) and (3.7b) provides the recurrence equation

$$\mathbf{P}_k = \mathbf{H}_k^{\mathrm{T}}\mathbf{H}_k + \mathbf{A}_k^{\mathrm{T}}\mathbf{P}_{k+1}[\mathbf{I} + (\mathbf{B}_k\mathbf{B}_k^{\mathrm{T}} - \gamma^{-2}\mathbf{D}_k\mathbf{D}_k^{\mathrm{T}})\mathbf{P}_{k+1}]^{-1}\mathbf{A}_k. \tag{3.8}$$

It is obvious that Eq. (3.8) is the same as the Riccati equation (2.6).

Besides constructing the solution of the Riccati equation, the conjoined basis which satisfies the boundary conditions, also plays a key role in the eigenvalue problem of the Hamiltonian difference system. A number γ^{-2} is said to be an eigenvalue of the Hamiltonian difference system

(3.1) and (3.4) if (3.1) has a non-trivial solution $(\mathbf{x}_k, \boldsymbol{\lambda}_k)$ which satisfies the general form of boundary conditions (3.4), and the solution is called an eigenfunction corresponding to the eigenvalue γ^{-2}. According to Ref. [2], if $(\mathbf{X}, \boldsymbol{\Lambda})$ is a conjoined basis of Eq. (3.1) with

$$\mathbf{X}_N = -\mathbf{R}_N^{\mathrm{T}}, \tag{3.9a}$$

$$\boldsymbol{\Lambda}_N = \mathbf{R}_N^{\#\mathrm{T}} \tag{3.9b}$$

then γ^{-2} is an eigenvalue of (3.1) and (3.4) if and only if the $n \times n$-matrix $\mathbf{R}_0^{\#}\mathbf{X}_0 + \mathbf{R}_0\boldsymbol{\Lambda}_0$ is singular. Comparing the boundary conditions (3.3) and (3.4) gives

$$\mathbf{X}_N = \mathbf{I}, \tag{3.10a}$$

$$\boldsymbol{\Lambda}_N = \mathbf{Q}_f, \tag{3.10b}$$

$$\mathbf{R}_0 = \mathbf{0}, \tag{3.11a}$$

$$\mathbf{R}_0^{\#} = \mathbf{I}. \tag{3.11b}$$

Therefore, the matrix $\mathbf{R}_0^{\#}\mathbf{X}_0 + \mathbf{R}_0\boldsymbol{\Lambda}_0 = \mathbf{X}_0$ is singular if and only if γ^{-2} is an eigenvalue of the Hamiltonian difference system (3.1) and (3.3). In this case, the solution of Riccati equation tends to infinity at $k = 0$ since $\mathbf{P}_0 = \boldsymbol{\Lambda}_0\mathbf{X}_0^{-1}$, which means the nonexistence of solution at $k = 0$ when γ^{-2} is an eigenvalue. In addition, just as the finite escape phenomena of H_∞-Riccati differential equations [1], when γ^{-2} is larger than the first order eigenvalue γ_1^{-2}, the determinant of matrices $\mathbf{M}_k$, of the solution to the Riccati equation (2.6) may change abruptly from positive to negative in $[0, N-1]$. But for $\gamma^{-2} = \gamma_1^{-2}$, the solution matrix $\mathbf{M}_k$ tends to infinity at $k = 0$ only. Since the critical value γ_{cr}^{-2} is the upper bound of γ^{-2} which ensures the existence of positive definite solution matrices to the Riccati difference equation (2.6), it is obvious that $\gamma_{cr}^{-2} = \gamma_1^{-2}$, as that in the case of continuous-time H_∞ optimization problems [4].

It is well known that eigenvalues of distributed systems are precisely stationary values of a Rayleigh quotient [10]. As shown in Refs. [3,4], the eigenvalues of a linear Hamiltonian differential system are stationary values of a generalized Rayleigh quotient with two kinds of variables. It is also easy to show that the eigenvalues of the Hamiltonian difference system (3.1) and (3.3) are stationary values of a generalized Rayleigh quotient of discrete form. The necessary condition of Eq. (3.12) having a stationary value with non-trivial solution is equivalent to the existence of a non-trivial solution to the Hamiltonian difference system (3.1) and (3.3).

$$\begin{aligned} J_\gamma(\mathbf{x}, \boldsymbol{\lambda}) = \sum_{k=0}^{N-1} \Big(& -\boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\mathbf{x}_{k+1} - \tfrac{1}{2}\boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\mathbf{B}_k\mathbf{B}_k^{\mathrm{T}}\boldsymbol{\lambda}_{k+1} \\ & + \tfrac{1}{2}\gamma^{-2}\boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\mathbf{D}_k\mathbf{D}_k^{\mathrm{T}}\boldsymbol{\lambda}_{k+1} + \boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\mathbf{A}_k\mathbf{x}_k + \tfrac{1}{2}\mathbf{x}_k^{\mathrm{T}}\mathbf{Q}_k\mathbf{x}_k \Big) + \tfrac{1}{2}\mathbf{x}_N^{\mathrm{T}}\mathbf{Q}_f\mathbf{x}_N. \end{aligned} \tag{3.12}$$

It is easy to show that the variational principle

$$\delta J_\gamma(\mathbf{x}, \boldsymbol{\lambda}) = 0 \tag{3.13}$$

leads to the generalized Rayleigh quotient

$$\gamma_j^{-2} = \mathrm{st}\frac{\Pi_1}{\Pi_2}, \quad j = 1, 2, 3, \ldots, \tag{3.14}$$

where

$$\Pi_1 = \sum_{k=0}^{N-1} \left(\boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\mathbf{x}_{k+1} - \boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\mathbf{A}_k\mathbf{x}_k - \tfrac{1}{2}\mathbf{x}_k^{\mathrm{T}}\mathbf{Q}_k\mathbf{x}_k + \tfrac{1}{2}\boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\mathbf{B}_k\mathbf{B}_k^{\mathrm{T}}\boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\right) - \tfrac{1}{2}\mathbf{x}_N^{\mathrm{T}}\mathbf{Q}_f\mathbf{x}_N, \tag{3.15a}$$

$$\Pi_2 = \tfrac{1}{2}\sum_{k=0}^{N-1} \boldsymbol{\lambda}_{k+1}^{\mathrm{T}}\mathbf{D}_k\mathbf{D}_k^{\mathrm{T}}\boldsymbol{\lambda}_{k+1}. \tag{3.15b}$$

The stationary values of the generalized Rayleigh quotient can be arranged as $\gamma_1^{-2} \leqslant \gamma_2^{-2} \leqslant \cdots$, i.e., $\gamma_1 \geqslant \gamma_2 \geqslant \cdots$. With Legendre transformation, it will be shown in next section that the generalized Rayleigh quotient stationary value (eigenvalue) problem (3.14) and (3.15) equals to a generalized matrix eigenvalue problem.

3.2. Relation between the generalized Rayleigh quotient and the matrix eigenvalue

Considering the generalized matrix eigenvalue problem

$$(\mathbf{K} - \rho\mathbf{M})\mathbf{d} = 0, \tag{3.16}$$

where the symmetric matrices $\mathbf{K} \geqslant 0$, $\mathbf{M} > 0$, and the vector $\mathbf{d} \in R^{Nn+n}$.

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{aa0} & \mathbf{K}_{ab0} & & & \\ & \mathbf{K}_{bb0} + \mathbf{K}_{aa1} & \mathbf{K}_{ab1} & & \\ & & \mathbf{K}_{bb1} + \mathbf{K}_{aa2} & & \\ & \textit{symmetry} & & \ddots & \mathbf{K}_{abN} \\ & & & & \mathbf{K}_{bbN} + \mathbf{Q}_f^{-1} \end{bmatrix}, \tag{3.17a}$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_0 & & & & \\ & \mathbf{M}_1 & & & \\ & & \mathbf{M}_2 & & \\ & & & \ddots & \\ & & & & \mathbf{M}_N \end{bmatrix}, \tag{3.17b}$$

$$\mathbf{d} = [\mathbf{d}_0^{\mathrm{T}}, \mathbf{d}_1^{\mathrm{T}}, \mathbf{d}_2^{\mathrm{T}}, \ldots, \mathbf{d}_N^{\mathrm{T}}]^{\mathrm{T}} \tag{3.17c}$$

in which $\mathbf{K}_{aai} \in R^{n\times n}$, $\mathbf{K}_{bbi} \in R^{n\times n}$ are symmetric matrices, $\mathbf{K}_{abi}^{\mathrm{T}} = \mathbf{K}_{bai} \in R^{n\times n}$, $\mathbf{d}_i \in R^n$ $(i = 0, 1, \ldots, N)$. It is well known that eigenvalues ρ_j of Eq. (3.16) are stationary values of the Rayleigh quotient

$$\rho_j = \mathrm{st}\frac{\mathbf{d}^{\mathrm{T}}\mathbf{K}\mathbf{d}}{\mathbf{d}^{\mathrm{T}}\mathbf{M}\mathbf{d}}, \quad j = 1, 2, 3, \ldots\ . \tag{3.18}$$

Considering the initial value of Eq. (2.1), let $\mathbf{d}_0 = 0$, then Eq. (3.18) becomes

$$\rho_j = \text{st}\, \frac{\sum_{k=0}^{N-1} U_k^0(\mathbf{d}_k, \mathbf{d}_{k+1}) + \frac{1}{2}\mathbf{d}_N^{\mathrm{T}}\mathbf{Q}_f^{-1}\mathbf{d}_N}{\sum_{k=0}^{N-1} \frac{1}{2}\mathbf{d}_{k+1}^{\mathrm{T}}\mathbf{M}_{k+1}\mathbf{d}_{k+1}}, \tag{3.19}$$

where

$$U_k^0(\mathbf{d}_k, \mathbf{d}_{k+1}) = \tfrac{1}{2}\mathbf{d}_k^{\mathrm{T}}\mathbf{K}_{aak}\mathbf{d}_k + \mathbf{d}_{k+1}^{\mathrm{T}}\mathbf{K}_{bak}\mathbf{d}_k + \tfrac{1}{2}\mathbf{d}_{k+1}^{\mathrm{T}}\mathbf{K}_{bbk}\mathbf{d}_{k+1}. \tag{3.20}$$

The eigenvalue problem (3.16) is equivalent to the variational problem

$$\delta[\tfrac{1}{2}\mathbf{d}^{\mathrm{T}}(\mathbf{K} - \rho\mathbf{M})\mathbf{d}] = 0, \tag{3.21}$$

i.e.,

$$\delta\left[\sum_{k=0}^{N-1} U_k^0(\mathbf{d}_k, \mathbf{d}_{k+1}) + \tfrac{1}{2}\mathbf{d}_N^{\mathrm{T}}\mathbf{Q}_f^{-1}\mathbf{d}_N - \sum_{k=0}^{N-1} \tfrac{1}{2}\rho\mathbf{d}_{k+1}^{\mathrm{T}}\mathbf{M}_{k+1}\mathbf{d}_{k+1}\right] = 0. \tag{3.22}$$

Denoting

$$U_k(\mathbf{d}_k, \mathbf{d}_{k+1}) = U_k^0(\mathbf{d}_k, \mathbf{d}_{k+1}) - \tfrac{1}{2}\rho\mathbf{d}_{k+1}^{\mathrm{T}}\mathbf{M}_{k+1}\mathbf{d}_{k+1}, \tag{3.23}$$

the variational equation (3.22) can be transformed into the canonical form by discrete Legendre transformation. Introducing

$$\mathbf{n}_k = \frac{\partial U_k}{\partial \mathbf{d}_k} = \mathbf{K}_{aak}\mathbf{d}_k + \mathbf{K}_{abk}\mathbf{d}_{k+1}, \tag{3.24a}$$

$$\mathbf{n}_{k+1} = -\frac{\partial U_k}{\partial \mathbf{d}_{k+1}} = -\mathbf{K}_{bbk}\mathbf{d}_{k+1} + \rho\mathbf{M}_{k+1}\mathbf{d}_{k+1} - \mathbf{K}_{bak}\mathbf{d}_k, \tag{3.24b}$$

then $U_k(\mathbf{d}_k, \mathbf{d}_{k+1})$ can be expressed as

$$U_k(\mathbf{d}_k, \mathbf{d}_{k+1}) = \tfrac{1}{2}\mathbf{n}_k^{\mathrm{T}}\mathbf{d}_k - \tfrac{1}{2}\mathbf{n}_{k+1}^{\mathrm{T}}\mathbf{d}_{k+1}. \tag{3.25}$$

Let $H_k(\mathbf{n}_k, \mathbf{d}_{k+1})$ denote the Hamiltonian function

$$H_k(\mathbf{n}_k, \mathbf{d}_{k+1}) = -\mathbf{d}_{k+1}^{\mathrm{T}}\mathbf{n}_{k+1} - U_k(\mathbf{d}_k, \mathbf{d}_{k+1}). \tag{3.26}$$

Solving for $\mathbf{n}_{k+1}$ and $\mathbf{d}_k$ from Eq. (3.24) gives the dual equations

$$\mathbf{n}_{k+1} = \mathbf{F}_k\mathbf{n}_k - \mathbf{G}_k\mathbf{d}_{k+1}, \tag{3.27a}$$

$$\mathbf{d}_k = \mathbf{E}_k\mathbf{n}_k + \mathbf{F}_k^{\mathrm{T}}\mathbf{d}_{k+1}, \tag{3.27b}$$

where

$$\mathbf{F}_k = -\mathbf{K}_{bak}\mathbf{K}_{aak}^{-1}, \tag{3.28a}$$

$$\mathbf{E}_k = \mathbf{K}_{aak}^{-1}, \tag{3.28b}$$

$$\mathbf{G}_k = \mathbf{K}_{bbk} - \mathbf{K}_{bak}\mathbf{K}_{aak}^{-1}\mathbf{K}_{abk} - \rho\mathbf{M}_{k+1} = \mathbf{G}_k^0 - \rho\mathbf{M}_{k+1}. \tag{3.28c}$$

Then the Hamiltonian function $H_k(\mathbf{n}_k, \mathbf{d}_{k+1})$ is given as

$$H_k(\mathbf{n}_k, \mathbf{d}_{k+1}) = -\tfrac{1}{2}\mathbf{n}_k^{\mathrm{T}}\mathbf{E}_k\mathbf{n}_k - \mathbf{d}_{k+1}^{\mathrm{T}}\mathbf{F}_k\mathbf{n}_k + \tfrac{1}{2}\mathbf{d}_{k+1}^{\mathrm{T}}\mathbf{G}_k\mathbf{d}_{k+1}. \tag{3.29}$$

Let $\mathbf{d}_N = \mathbf{Q}_f\mathbf{n}_N$, the canonical form of the variational problem (3.22) is

$$\delta\left[\sum_{k=0}^{N-1}(-\mathbf{n}_{k+1}^{\mathrm{T}}\mathbf{d}_{k+1} - H_k(\mathbf{n}_k, \mathbf{d}_{k+1})) + \tfrac{1}{2}\mathbf{n}_N^{\mathrm{T}}\mathbf{Q}_f\mathbf{n}_N\right]$$
$$= \delta\left[\sum_{k=0}^{N-1}\left(-\mathbf{d}_{k+1}^{\mathrm{T}}\mathbf{n}_{k+1} + \mathbf{d}_{k+1}^{\mathrm{T}}\mathbf{F}_k\mathbf{n}_k + \tfrac{1}{2}\mathbf{n}_k^{\mathrm{T}}\mathbf{E}_k\mathbf{n}_k - \tfrac{1}{2}\mathbf{d}_{k+1}^{\mathrm{T}}\mathbf{G}_k\mathbf{d}_{k+1}\right) + \tfrac{1}{2}\mathbf{n}_N^{\mathrm{T}}\mathbf{Q}_f\mathbf{n}_N\right]. \quad (3.30)$$

Therefore, the variational principle (3.19) can be transformed to be an equivalent generalized Rayleigh quotient with two kinds of variables

$$\rho_j = st\frac{\Phi_1}{\Phi_2}, \quad (3.31)$$

where

$$\Phi_1 = \sum_{k=0}^{N-1}\left(\mathbf{d}_{k+1}^{\mathrm{T}}\mathbf{n}_{k+1} - \mathbf{d}_{k+1}^{\mathrm{T}}\mathbf{F}_k\mathbf{n}_k - \tfrac{1}{2}\mathbf{n}_k^{\mathrm{T}}\mathbf{E}_k\mathbf{n}_k + \tfrac{1}{2}\mathbf{d}_{k+1}^{\mathrm{T}}\,\mathbf{G}_k^0\mathbf{d}_{k+1}\right) - \tfrac{1}{2}\mathbf{n}_N^{\mathrm{T}}\mathbf{Q}_f\mathbf{n}_N, \quad (3.32a)$$

$$\Phi_2 = \sum_{k=0}^{N-1}\tfrac{1}{2}\mathbf{d}_{k+1}^{\mathrm{T}}M_{k+1}\mathbf{d}_{k+1}. \quad (3.32b)$$

Eqs. (3.31) and (3.14) have the identical formulation, and the algorithm for the search of the fundamental eigenvalue of Eq. (3.31) can also be used for the computation of γ_{cr}^{-2}. Since $\mathbf{d}\in R^{Nn+n}$, the dimension of the eigenvalue problem (3.16) may be very large. But the H_∞ control problem needs only the fundamental eigenvalue, so it is reasonable to consider a special algorithm for the eigenvalue computation. As mentioned in Section 1, the W–W algorithm can only be applied to the eigenvalue problem of one kind of variables, such as Eq. (3.16) or (3.18). For the eigenvalue problems such as Eq. (3.31), the W–W algorithm should be extended [3,4].

4. Algorithm for critical value computation

4.1. The Wittrick–Williams algorithm

The W–W algorithm for the computation of eigenvalues is based on the eigenvalue count of a specified interval [5]. Consider the eigenvalue problem

$$(\mathbf{A} - \rho\mathbf{B})\mathbf{d} = \mathbf{0}, \quad (4.1)$$

where $\mathbf{d}\in R^n$, $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$, and $\mathbf{B}^{\mathrm{T}} = \mathbf{B}$ being positive definite. The number of eigenvalues, which are less than the given value $\rho_\#$, is defined as the *eigenvalue count* of the eigenvalue problem (4.1). Let $J(\rho_\#)$ denote this number, then

$$J(\rho_\#) = s\{\mathbf{A} - \rho_\#\mathbf{B}\}, \quad (4.2)$$

where $s\{\mathbf{C}\}$ denotes the number of negative eigenvalues of the symmetric matrix $\mathbf{C}$, note that $s\{\mathbf{C}\} = s\{\mathbf{C}^{-1}\}$.

If the matrices and vector are partitioned as

$$\left\{\begin{bmatrix}\mathbf{A}_{aa} & \mathbf{A}_{ab}\\ \mathbf{A}_{ba} & \mathbf{A}_{bb}\end{bmatrix} - \rho\begin{bmatrix}\mathbf{B}_{aa} & \mathbf{B}_{ab}\\ \mathbf{B}_{ba} & \mathbf{B}_{bb}\end{bmatrix}\right\}\begin{bmatrix}\mathbf{d}_a\\ \mathbf{d}_b\end{bmatrix} = \mathbf{0}, \tag{4.3}$$

where $\mathbf{d}_a \in R^r$, $\mathbf{d}_b \in R^{n-r}$. And r constraints are given as $\mathbf{d}_a = \mathbf{0}$, then the eigenvalue problem is reduced to be

$$(\mathbf{A}_{bb} - \rho\mathbf{B}_{bb})\mathbf{d}_b = \mathbf{0}. \tag{4.4}$$

Let $J_0(\rho_\#)$ be the eigenvalue count of Eq. (4.4), and denote the reduced matrix as

$$\mathbf{D}(\rho_\#) = \mathbf{A}_{aa} - \rho_\#\mathbf{B}_{aa} - (\mathbf{A}_{ab} - \rho_\#\mathbf{B}_{ab})(\mathbf{A}_{bb} - \rho_\#\mathbf{B}_{bb})^{-1}(\mathbf{A}_{ba} - \rho_\#\mathbf{B}_{ba}). \tag{4.5}$$

Then, the eigenvalue count of Eq. (4.1) is

$$J(\rho_\#) = J_0(\rho_\#) + s\{\mathbf{D}(\rho_\#)\}. \tag{4.6}$$

For an eigenvalue problem of continuum, the degrees of freedom $n \to \infty$, and $\mathbf{D}(\rho)$ becomes a transcendental eigenvalue problem of ρ, for which Eq. (4.6) still holds [5,6].

4.2. The extended Wittrick–Williams algorithm

To simplify the expression, introducing the *dynamic stiffness matrix* $(\mathbf{A} - \rho\mathbf{B}) = \mathbf{C}(\rho)$ and the eigenvalue problem is written as $\mathbf{Cd} = \mathbf{0}$.

Consider the eigenvalue problem of a segment of Eqs. (3.16) and (3.17)

$$\begin{bmatrix}\mathbf{C}_{aa1} & \mathbf{C}_{ab1} & \\ \mathbf{C}_{ba1} & \mathbf{C}_{aa2} + \mathbf{C}_{bb1} & \mathbf{C}_{ab2}\\ & \mathbf{C}_{ba2} & \mathbf{C}_{bb2}\end{bmatrix}\begin{bmatrix}\mathbf{d}_{k-1}\\ \mathbf{d}_k\\ \mathbf{d}_{k+1}\end{bmatrix} = \mathbf{0}, \tag{4.7}$$

where

$$\mathbf{C}_{aa1} = \mathbf{K}_{aak} - \rho\mathbf{M}_{aak}, \quad \mathbf{C}_{aa2} = \mathbf{K}_{aa,k+1} - \rho\mathbf{M}_{aa,k+1},$$

$$\mathbf{C}_{ab1} = \mathbf{K}_{abk} - \rho\mathbf{M}_{abk}, \quad \mathbf{C}_{ba1} = \mathbf{K}_{bak} - \rho\mathbf{M}_{bak},$$

$$\mathbf{C}_{ab2} = \mathbf{K}_{ab,k+1} - \rho\mathbf{M}_{ab,k+1}, \quad \mathbf{C}_{ba2} = \mathbf{K}_{ba,k+1} - \rho\mathbf{M}_{ba,k+1},$$

$$\mathbf{C}_{bb1} = \mathbf{K}_{bbk} - \rho\mathbf{M}_{bbk}, \quad \mathbf{C}_{bb2} = \mathbf{K}_{bb,k+1} - \rho\mathbf{M}_{bb,k+1}.$$

All the eigenvalue problems in this section are expressed in terms of dynamic stiffness matrix. According to Eqs. (3.28a)–(3.28c), let

$$\mathbf{F}_i = -\mathbf{C}_{bai}\mathbf{C}_{aai}^{-1}, \quad i = 1, 2, \tag{4.8a}$$

$$\mathbf{E}_i = \mathbf{C}_{aai}^{-1}, \tag{4.8b}$$

$$\mathbf{G}_i = \mathbf{C}_{bbi} - \mathbf{C}_{bai}\mathbf{C}_{aai}^{-1}\mathbf{C}_{abi}. \tag{4.8c}$$

Then

$$\mathbf{C}_{bai} = -\mathbf{F}_i\mathbf{E}_i^{-1}, \tag{4.9a}$$

$$\mathbf{C}_{aai} = \mathbf{E}_i^{-1}, \tag{4.9b}$$

$$\mathbf{C}_{bbi} = \mathbf{G}_i + \mathbf{F}_i\mathbf{E}_i^{-1}\mathbf{F}_i^{\mathrm{T}}, \tag{4.9c}$$

where $i = 1, 2$. Define for the segments k and $k+1$

$$\mathbf{n}_k = \mathbf{F}_1\mathbf{n}_{k-1} - \mathbf{G}_1\mathbf{d}_k, \tag{4.10a}$$

$$\mathbf{d}_{k-1} = \mathbf{E}_1\mathbf{n}_{k-1} + \mathbf{F}_1^{\mathrm{T}}\mathbf{d}_k, \tag{4.10b}$$

$$\mathbf{n}_{k+1} = \mathbf{F}_2\mathbf{n}_k - \mathbf{G}_2\mathbf{d}_{k+1}, \tag{4.10c}$$

$$\mathbf{d}_k = \mathbf{E}_2\mathbf{n}_k + \mathbf{F}_2^{\mathrm{T}}\mathbf{d}_{k+1} \tag{4.10d}$$

and for the combined segment

$$\mathbf{n}_{k+1} = \mathbf{F}_c\mathbf{n}_{k-1} - \mathbf{G}_c\mathbf{d}_{k+1}, \tag{4.11a}$$

$$\mathbf{d}_{k-1} = \mathbf{E}_c\mathbf{n}_{k-1} + \mathbf{F}_c^{\mathrm{T}}\mathbf{d}_{k+1}. \tag{4.11b}$$

The following matrix merging equations are derived [7]:

$$\mathbf{G}_c = \mathbf{G}_2 + \mathbf{F}_2(\mathbf{G}_1^{-1} + \mathbf{E}_2)^{-1}\mathbf{F}_2^{\mathrm{T}}, \tag{4.12a}$$

$$\mathbf{E}_c = \mathbf{E}_1 + \mathbf{F}_1^{\mathrm{T}}(\mathbf{E}_2^{-1} + \mathbf{G}_1)^{-1}\mathbf{F}_1, \tag{4.12b}$$

$$\mathbf{F}_c = \mathbf{F}_2(\mathbf{I} + \mathbf{G}_1\mathbf{E}_2)^{-1}\mathbf{F}_1. \tag{4.12c}$$

Let the constraint be denoted as $\mathbf{d}_{k+1} = 0$, the eigenvalue problem (4.7) is reduced to

$$\begin{bmatrix} \mathbf{C}_{aa1} & \mathbf{C}_{ab1} \\ \mathbf{C}_{ba1} & \mathbf{C}_{aa2} + \mathbf{C}_{bb1} \end{bmatrix} \begin{bmatrix} \mathbf{d}_{k-1} \\ \mathbf{d}_k \end{bmatrix} = \mathbf{0}. \tag{4.13}$$

Since the identity

$$\begin{bmatrix} \mathbf{C}_{aa1} & \mathbf{C}_{ab1} \\ \mathbf{C}_{ba1} & \mathbf{C}_{bb1} + \mathbf{C}_{aa2} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{C}_{ba1}\mathbf{C}_{aa1}^{-1} & \mathbf{I} \end{bmatrix} \times \begin{bmatrix} \mathbf{C}_{aa1} & \mathbf{C}_{ab1} \\ \mathbf{0} & \mathbf{C}_{aa2} + \mathbf{C}_{bb1} - \mathbf{C}_{ba1}\mathbf{C}_{aa1}^{-1}\mathbf{C}_{ab1} \end{bmatrix} \tag{4.14}$$

is valid for any given $\rho_\#$, the eigenvalue count of Eq. (4.13) is

$$J_{Rc}(\rho_\#) = s\{\mathbf{C}_{aa2} + \mathbf{C}_{bb1} - \mathbf{C}_{ba1}\mathbf{C}_{aa1}^{-1}\mathbf{C}_{ab1}\} + s\{\mathbf{C}_{aa1}\} = s\{\mathbf{G}_1 + \mathbf{E}_2^{-1}\} + s\{\mathbf{E}_1^{-1}\}. \tag{4.15}$$

Let $J_{R1}(\rho_\#) = s\{\mathbf{K}_{aa1}\}$, $J_{R2}(\rho_\#) = s\{\mathbf{K}_{aa2}\}$, then

$$J_{Rc}(\rho_\#) = J_{R1}(\rho_\#) + J_{R2}(\rho_\#) - s\{\mathbf{E}_2\} + s\{\mathbf{G}_1 + \mathbf{E}_2^{-1}\}. \tag{4.16}$$

Executing this equation repeatedly with the merging equation (4.12) derives the eigenvalue count of eigenvalue problem (3.31), which corresponds to the case of no boundary constraint.

It is easy to show that the eigenvalue count of the eigenvalue problem

$$\begin{bmatrix} \mathbf{C}_{aa2} & \mathbf{C}_{ab2} \\ \mathbf{C}_{ba2} & \mathbf{C}_{bb2} + \mathbf{Q}_f^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{d}_k \\ \mathbf{d}_{k+1} \end{bmatrix} = \mathbf{0} \tag{4.17}$$

is

$$J_{RQ2}(\rho_\#) = s\{\mathbf{Q}_f^{-1} + \mathbf{G}_2\} + s\{\mathbf{C}_{aa2}\} = J_{R2}(\rho_\#) + s\{\mathbf{Q}_f^{-1} + \mathbf{G}_2\}. \tag{4.18}$$

Now consider the following eigenvalue problem:

$$\begin{bmatrix} \mathbf{C}_{aa1} & \mathbf{C}_{ab1} & \\ \mathbf{C}_{ba1} & \mathbf{C}_{aa2} + \mathbf{C}_{bb1} & \mathbf{C}_{ab2} \\ & \mathbf{C}_{ba2} & \mathbf{C}_{bb2} + \mathbf{Q}_f^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{d}_{k-1} \\ \mathbf{d}_k \\ \mathbf{d}_{k+1} \end{bmatrix} = \mathbf{0}. \tag{4.19}$$

Let $\mathbf{d}_{k+1} = \mathbf{0}$, the eigenvalue count of the following eigenvalue problem:

$$\begin{bmatrix} \mathbf{C}_{aa1} & \mathbf{C}_{ab1} \\ \mathbf{C}_{ba1} & \mathbf{C}_{aa2} + \mathbf{C}_{bb1} \end{bmatrix} \begin{bmatrix} \mathbf{d}_{k-1} \\ \mathbf{d}_k \end{bmatrix} = \mathbf{0}, \tag{4.20}$$

is

$$J_0(\rho_\#) = J_{Rc}(\rho_\#). \tag{4.21}$$

According to Eq. (4.5), $\mathbf{D}(\rho_\#)$ of this problem is

$$\mathbf{D}(\rho_\#) = (\mathbf{Q}_f^{-1} + \mathbf{C}_{bb2}) - \begin{bmatrix} \mathbf{0} & \mathbf{C}_{ba2} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{aa1} & \mathbf{C}_{ab1} \\ \mathbf{C}_{ba1} & \mathbf{C}_{bb1} + \mathbf{C}_{aa2} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{C}_{ab2} \end{bmatrix}. \tag{4.22}$$

Using the matrix inversion lemma and Eq. (4.12a) gives

$$\mathbf{D}(\rho_\#) = \mathbf{Q}_f^{-1} + \mathbf{G}_2 + \mathbf{F}_2(\mathbf{I} + \mathbf{G}_1\mathbf{E}_2)^{-1}\mathbf{G}_1\mathbf{F}_2^{\mathrm{T}} = \mathbf{Q}_f^{-1} + \mathbf{G}_c. \tag{4.23}$$

The final eigenvalue count equation of Eq. (4.19) is

$$J_{RQc}(\rho_\#) = J_{Rc}(\rho_\#) + s\{\mathbf{Q}_f^{-1} + \mathbf{G}_c\}. \tag{4.24}$$

Executing this equation repeatedly with the interval merging Eq. (4.12) derives the eigenvalue count of the eigenvalue problem (3.31).

4.3. Procedure of critical value computation

This section describes the computational procedure for time-invariant systems. The procedure can be readily generalized to time-variant systems according to Section 4.2.

Finite horizon case

0. {Select a suitable $\gamma_\#^{-2}$; $\mathbf{G} = \mathbf{BB}^{\mathrm{T}} - \gamma^{-2}\mathbf{DD}^{\mathrm{T}}$; $\mathbf{F} = \mathbf{A}$; $\mathbf{E} = \mathbf{H}^{\mathrm{T}}\mathbf{H}$},
1. {$\mathbf{E}_1 = \mathbf{E}$; $\mathbf{G}_1 = \mathbf{G}$; $\mathbf{F}_1 = \mathbf{F}$; $J_{R1} = 0$; $\mathbf{E}_2 = \mathbf{Q}_f$; $\mathbf{G}_2 = \mathbf{0}$; $\mathbf{F}_2 = \mathbf{I}$; $J_{R2} = 0$},
2. {for ($k = 1$; $k \leqslant N-1$; $k++$)

```
    {
        {Compute E_c, G_c, F_c and J_RQc from (4.12a) to (4.12c) and (4.16)}
        {E_2 = E_c; G_2 = G_c; F_2 = F_c; J_RQ2 = J_RQc}
        if (J_RQc > 0)
        {γ_#^-2 is an upper bound (ub) of γ_cr^-2, jump out of loop
                and restart from step 0 with lower γ_#^-2}
    }
```

3. {Now γ^{-2} is a sub-optimal parameter and is a lower bound (*lb*) of γ_{cr}^{-2}}
   if $(ub-lb) > \varepsilon_1 (\varepsilon_1 > 0)$
   {increase γ^{-2} and restart from step 0}
   else
   {break}.

The iteration for $\gamma_{\#}^{-2}$ should be continued until the specified precision is reached, i.e. $(\text{ub}-\text{lb}) < \varepsilon_1$ is achieved. The lower bound is taken as γ_{cr}^{-2}. It should be noted that the sequence $\mathbf{E}_c$ generated by the above procedure is just the sequence $\mathbf{M}_k$, i.e., the solution of Eq. (2.6) or (2.12) [7]. As N approaching infinity, $\mathbf{M}_0$ becomes the solution of the Riccati algebraic equation (2.14) of the infinite-horizon case [1,8].

Infinite horizon case

1. {Select a suitable γ^{-2}; $\mathbf{F}_c = \mathbf{A}$; $\mathbf{G}_c = \mathbf{B}\mathbf{B}^{\mathrm{T}} - \gamma^{-2}\mathbf{D}\mathbf{D}^{\mathrm{T}}$; $\mathbf{E}_c = \mathbf{H}^{\mathrm{T}}\mathbf{H}$; $J_{Rc} = 0$},
2. {while ($\|\mathbf{F}_c\| > \varepsilon_2$) ($\varepsilon_2 > 0, \|\mathbf{F}_c\|$ is the 2-norm of matrix $\mathbf{F}_c$)}
   {
   {$\mathbf{E}_1 = \mathbf{E}_2 = \mathbf{E}_c$; $\mathbf{G}_1 = \mathbf{G}_2 = \mathbf{G}_c$; $\mathbf{F}_1 = \mathbf{F}_2 = \mathbf{F}_c$; $J_{R1} = J_{R2} = J_{Rc}$}
   {Compute $\mathbf{E}_c$, $\mathbf{G}_c$, $\mathbf{F}_c$ and J_{Rc} from Eqs. (4.12a) to (4.12c), (4.16)}
   if ($J_{Rc} > 0$)
   {Jump out of the loop; restart from step 1 with a lower γ^{-2}}
   }

At the end of the iteration, $\mathbf{E}_c$ equals to the solution of the Riccati algebraic equation (2.14) of the infinite horizon control problem, which is also the stable solution of the Riccati equation (2.6) or (2.12).

5. Examples

Example 1. For the convenience of comparing, this example is taken from Ref. [11]. The data of a discrete time system are

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -0.1 & 0 \\ 0 & 1 & 0 & -0.1 \\ 0.033 & -0.033 & 1 & 0 \\ 0.033 & -0.033 & -0.007 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -0.004 & 0 \\ 0 & -0.004 \\ 0.085 & 0 \\ 0 & 0.085 \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ -2.113 & 2.113 & 0.375 & 0.375 \end{bmatrix}.$$

The critical values of the Riccati equation of finite horizon of different length are shown in Table 1. As N tends to infinity, the critical value of the associated Riccati difference equation approaches the critical value of the Riccati algebraic equation of infinite horizon case. It should be noted that even when N increases from 2^8 to 2^{10}, the amount of calculation increases just a little.

Table 1
Critical value γ_{cr} of Example 1

N	2^2	2^3	2^5	2^8	2^{10}
γ_{cr}	0.13803	0.15944	0.18870	0.18884	0.18884

Table 2
Critical value γ_{cr} of Example 2

N	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
γ_{cr}	8.5082	8.5748	8.6235	8.8429	17.100	20.217	20.237	20.237	20.237

According to Table 1, the critical value is $\gamma_{cr} \approx 0.18884$ when N is large enough. One can also obtain the same value of γ_{cr} by using of the algorithm of the infinite horizon case. Furthermore, the algorithm also gives the solution of the Riccati equation. When $\gamma = 0.25$ and $N = 2^8$, the solution of the Riccati difference equation at $k = 0$ is

$$\mathbf{M} = \begin{bmatrix} 73.198 & -73.786 & -42.565 & 32.735 \\ & 96.810 & 39.473 & -48.329 \\ & \mathit{symmetry} & 38.820 & -22.316 \\ & & & 40.685 \end{bmatrix},$$

which is also the solution of the algebraic Riccati equation. According to Eq. (2.7a), the state feedback gain matrix is

$$\begin{aligned} \mathbf{K} &= -\mathbf{B}^{\mathrm{T}}\mathbf{P}[\mathbf{I} + (\mathbf{B}\mathbf{B}^{\mathrm{T}} - \gamma^{-2}\mathbf{D}\mathbf{D}^{\mathrm{T}})\mathbf{M}]^{-1}\mathbf{A} \\ &= \begin{bmatrix} 3.226 & -2.952 & -3.086 & 1.739 \\ -2.536 & 3.762 & 1.732 & -3.228 \end{bmatrix}, \end{aligned}$$

which is identical to the results in Ref. [11].

Example 2. The data of the second discrete time system are

$$\mathbf{A} = \begin{bmatrix} 1.0 & -32.37 & 0.0 & 32.2 & 0.0 & 0.7 & 0.0 \\ -0.00014 & 1.0 & 10.0 & 0.0 & 2.0 & 0.0 & 0.3 \\ -0.0111 & -34.72 & 1.0 & 0.0 & 0.0 & 1.4 & 0.0 \\ 0.0 & 0.0 & 1.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 1.7 \\ 0.0 & -1.0 & 0.1 & 3.2 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.1 & -1.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0.0 \\ 0.0 \\ 0.2 \\ 0.0005 \\ 0.1 \\ 0.02 \\ 0.01 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0.0 \\ -0.001064 \\ -0.338 \\ 0.0 \\ 0.2 \\ 0.1 \\ 0.01 \end{bmatrix},$$

$$\mathbf{H} = \mathrm{diag}(0.5, 0.5, 0.5, 1.0, 1.0, 1.0).$$

The critical values of the associated Riccati equation of different finite horizon are shown in Table 2. As N tends to infinity, the critical value tends to a constant, which is also the critical value of the associated Riccati algebraic equation.

6. Conclusions

The formulation of the extended W–W algorithm fit for the critical value computation of the H_∞-Riccati difference equation of discrete-time H_∞ control system is presented. Which is based on the equivalence between γ_{cr}^{-2} and the first order eigenvalue of the associated Hamiltonian difference system. The generalized Rayleigh quotient and discrete Legendre transformation bridge the linear Hamiltonian eigenvalue problem and the matrix eigenvalue problem. Based on this relation, some key computational issues of discrete time H_∞ filtering and H_∞ measurement feedback control problems can also be solved by the algorithm.

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